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# SAFETY ANALYSIS OF STOCHASTIC FINITE ELEMENT SYSTEMS BY MONTE CARLO SIMULATION

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Probabilistic safety analysis methods for structural systems involving spatially variable material properties are studied. For the cases to which the first-order second-moment methods cannot be applied well, two different Monte Carlo methods are employed in conjunction with the finite element technique. "Monte Carlo simulation", which is based on the stochastic field simulation technique and the safety judgment for each trial, is recommended because its accuracy is guaranteed irrespective of the number of random variables. On the contrary, the accuracy of "Monte Carlo integration", where a numerical integration of the joint density function of the random variables is carried out by means of sample-mean Monte Carlo method, goes down dramatically as the number of random variables getting large. Numerical examples verify the above argument and reveal the effects of correlation parameters on the local and system limit state probabilities.

*Keywords : probabilistic safety analysis, stochastic fields, finite element analysis, Monte Carlo methods*

## 1. INTRODUCTION

Methods of probabilistic safety analysis have been developed and extensively studied by a number of researches in the last two decades or so. A primary measure of safety has been considered to be the limit state probability, $P_f$, which can be obtained by the integration of the joint probability density function over the domain where the limit state condition is satisfied. However, such a direct evaluation of $P_f$ often presents difficulties in constructing the joint probability density function of random variables involved and in performing multi-dimensional integration over the domain which is very often irregular. In order to circumvent such difficulties, the first-order second-moment method has been developed where only expected values and variances are used to make the safety statement. Because the first-order second-moment method suffers from the invariance problem, the so-called advanced first-order second-moment method has been further developed[1]. The advanced first-order second-moment method enables one to obtain a safety index which has a precise mathematical meaning : the minimum distance from the origin to the transformed limit state surface, by an iterative procedure[2] or the method of Lagrange multipliers[3].

However, the structural models to which such safety analysis methods have been applied are mostly simple ones containing at most several random variables. Also, the load effects and resisting capacities of those models are represented by explicit functions of the basic random variables whose probabilistic characteristics are given. The recent development of the computer science, however, enabled us to use more sophisticated structural modeling technique, notably finite element methods. In this connection, the perturbation and related methods[e.g.,4)~6)] are applied to obtain the reliability index of finite element

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systems. However, when the number of random variables becomes considerably large, such analytical approaches tend to be of limited value. Even if the safety index for a failure mode is obtained, rigorous evaluation of the limit state probability from the safety index is almost impossible, since the limit state surface is mostly irregular and the distribution of the load-effects are unknown. Also, the system limit state probability, which may be defined as a function of several correlated failure modes or defined by a union or an intersection of several local failure events, is no more easy to obtain.

In view of this, Monte Carlo methods are considered as an alternative for the safety evaluation of finite element systems involving stochastic fields. Although Monte Carlo methods are often followed by a heavy computational effort, analytical methods are sometimes even more costly to obtain the same amount of information if not impossible. Currently, Monte Carlo methods may be the only practical and reliable tool for evaluating the structural safety of complicated structural systems. The application of Monte Carlo methods to finite element systems with spatially variable material properties has been presented by the current authors[7),8)]. As an extension of such studies to structural safety assessment, two different Monte Carlo methods, "Monte Carlo simulation" and "Monte Carlo integration", are compared especially with respect to the accuracy, and their numerical examples and discussion are provided.

## 2. ANALYTICAL MODELING AND SIMULATION OF STOCHASTIC VECTORS

The spatial variation of material properties is assumed to be a multi-variate multi-dimensional homogeneous stochastic field. If a material property, $x_i$, is log-normally distributed, $x_i$ can be transformed into its standardized form as

$$X_i=(\ln x_i-\ln \breve{x}_i)/\zeta_i \quad \text{or} \quad x_i=\breve{x}_i \exp(X_i\zeta_i) \cdots\cdots (1)$$

in which $\breve{x}_i$ and $\zeta_i$ are the median and logarithmic standard deviation of $x_i$, respectively. After this standardization, the stochastic field of $X_i$ is Gaussian and mean-zero : $E[X_i]=0$. The spatial variability of such standardized material properties $X$'s is defined by their auto-correlation (when $i=j$) and cross-correlation (when $i\neq j$) functions :

$$R_{X_iX_j}(\boldsymbol{\xi})=E[X_i(\boldsymbol{x})X_j(\boldsymbol{x}+\boldsymbol{\xi})] \text{ with } \boldsymbol{x}=[xy\cdots]^T \text{ and } \boldsymbol{\xi}=[\xi_x\xi_y\cdots]^T \cdots\cdots (2)$$

where $\boldsymbol{x}$ is the position vector and $\boldsymbol{\xi}$ the separation vector of an appropriate dimension. With the aid of the Wiener-Khintchine relationship, corresponding power and cross-spectral density functions of the stochastic field can be obtained. In the present study, the following auto- and cross-correlation functions for bi-variate two-dimensional stochastic fields are employed for numerical examples.

$$R_{X_1X_1}(\boldsymbol{\xi})=R_{X_2X_2}(\boldsymbol{\xi})=\exp\left[-\left(\frac{\xi_x}{d_x}\right)^2-\left(\frac{\xi_y}{d_y}\right)^2\right]; \quad R_{X_1X_2}(\boldsymbol{\xi})=R_{X_2X_1}(\boldsymbol{\xi})=\beta\exp\left[-\left(\frac{\xi_x}{d_x}\right)^2-\left(\frac{\xi_y}{d_y}\right)^2\right]$$
$$\cdots\cdots (3\cdot a, b)$$

in which $d_x$ and $d_y$ are parameters such that the larger they are, the more slowly the correlation disappears, and $\beta$ is a parameter indicating the rate of cross-correlation. It is mentioned that, in practical applications[9)], the form of correlation functions is less important than the values of their parameters.

As the method of stochastic field simulation, the modal decomposition method is employed among other candidates, such as spectral representation technique[10),11)] and ARMA models[12)], while these other methods are also quite possible to apply to the analysis hereafter. In the modal decomposition method, a structure is firstly divided into an appropriate number of finite elements. The size of each finite element must be small enough from the spatial variability of the material properties as well as strain and stress gradients points of view. From the former point of view, it must be small enough so that the property values can be considered approximately constant within each element. Hence, if there are $n$ finite elements in total and $m$ material property parameters which exhibit random variability, then there are $N=n\times m$ material property values. Thus, the stochastic field is reduced to a vector $\boldsymbol{X}$ of $N$-dimension as follows :

$$\boldsymbol{X}=[\{\boldsymbol{X}^{(1)}\}^T\{\boldsymbol{X}^{(2)}\}^T\cdots\{\boldsymbol{X}^{(m)}\}^T]^T \text{ with } \boldsymbol{X}^{(i)}=[X_1^{(i)}X_2^{(i)}\cdots\cdots X_n^{(i)}]^T \quad (i=1,2,\cdots,m) \cdots\cdots (4)$$

The correlational characteristics of the stochastic vector $\boldsymbol{X}$ can be specified in terms of its covariance

matrix :

$$C_{XX}=\begin{bmatrix} C_{XX}^{11} & C_{XX}^{12} & \cdots & C_{XX}^{1m} \\ C_{XX}^{21} & C_{XX}^{22} & \cdots & C_{XX}^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ C_{XX}^{m1} & C_{XX}^{m2} & \cdots & C_{XX}^{mm} \end{bmatrix} \quad (5)$$

From a cross-correlation function between the zero-mean stochastic vectors $X^{(i)}$ and $X^{(j)}$, the $kl$-component of the above submatrix $C_{XX}^{ij}$ is determined by

$$\mathrm{cov}[X_k^{(i)}, X_l^{(j)}]=R_{X_iX_j}(\xi_{kl}) \quad (6)$$

with $\xi_{kl}$ being the separation vector between the centroids of elements $k$ and $l$.

The covariance matrix $C_{XX}$ is always symmetric regardless of the assumed form of correlation functions. Hence, the real eigenvalues and eigenvectors of $C_{XX}$ are obtained by solving the following eigenequation :

$$C_{XX}\Phi_X=\Phi_X\Lambda_X \text{ with } \Phi_x=[\phi_1\phi_2\cdots\phi_N]^T \quad (7)$$

in which $\Phi_X$ is the modal matrix whose column vectors are orthogonal each other and have Euclidean lengths of 1 as $\Phi_X^T\Phi_X=I$, and $\Lambda_X$ is the diagonal eigenvalue matrix whose diagonal members are numbered as follows : $|\lambda_1|\geqq|\lambda_2|\geqq\cdots|\lambda_N|$. It is not easy to generate a stochastic vector $X$, whose components are correlated each other, directly. Thus, a stochastic vector $Z=[Z_1Z_2\cdots Z_N]^T$, whose components are Gaussian and independent each other and have zero-means and variances $\lambda_i$ $(i=1, 2, \cdots, N)$, is generated as a first stage. Then, the correlated stochastic vector $X$ is obtained by the transformation : $X=\Phi_XZ$. $X$ is also Gaussian since it is represented by the linear combination of Gaussian random variables. If $X$ exhibits non-Gaussian characteristics, the simulation technique for non-Gaussian fields[13] can be applied. The generated samples of $X$ satisfy the target covariance matrix in ensemble sense. Considering the decaying feature of the eigenvalues, $X$ may be approximated by its truncated form using only the first $M$ $(M\leqq N)$ significant modes and corresponding independent random variables as

$$X=\Phi_XZ\approx[\phi_1\phi_2\cdots\phi_m][Z_1Z_2\cdots Z_M]^T \quad (8)$$

The error caused by the truncation can be evaluated by the normalized sum of eigenvalues up to the $M$-th mode as

$$P_\lambda(M)=\left(\sum_{k=1}^{M}\lambda_k\right)\Big/\left(\sum_{k=1}^{N}\lambda_k\right)=\sum_{k=1}^{M}\lambda_k/N \quad (9)$$

The summation of all the eigenvalues represents the total power of variation of the stochastic vector. Thus, $P_\lambda(M)$ indicates the accuracy of generated samples after the truncation. $M$ can usually be taken much smaller than $N$, since the eigenvalues of higher-order modes are negligibly small except for white noise stochastic fields. Such a truncation of modes may contribute for reducing the sample size and CPU time for Monte Carlo methods[14]

## 3. MONTE CARLO SIMULATION

The limit state of a structure modeled by finite elements under various loading conditions can generally be described by

$$g(X)=g(X_1, X_2, \cdots, X_N)\leqq 0 \quad (10)$$

where $g(X)=0$ is known as a limit state surface, $X$ indicates a set of spatially variable parameters (stochastic vector) of the system, and $g(\cdot)$ is the function of the load effects (e. g., stress and strain) obtained by the finite element analysis. Following the stochastic vector simulation procedure, the realization of the stochastic vector $X$ can be obtained. If the limit state condition for the $k$-th finite element is given by $g_k(X)\leqq 0$, one can examine if $g_k(X^{(j)})\leqq 0$ is satisfied for $j=1, 2, \cdots, N_s$ by means of Monte Carlo simulation. Here, $X^{(j)}$ and $N_s$ indicate the $j$-th sample of $X$ and the sample size, respectively. The element (local) limit state probability is then estimated by introducing performance function $\delta_k$ for safety judgment.

$$\text{Prob}(E_k) \approx P_f^{(k)} = \frac{1}{N_s} \sum_{j=1}^{N_s} \delta_k(\boldsymbol{X}^{(j)}); \quad \delta_k(\boldsymbol{X}^{(j)}) = \begin{cases} 1 & \text{if } g_k(X^{(j)}) \leqq 0 \\ 0 & \text{otherwise} \end{cases} \quad \cdots\cdots (11\cdot a, b)$$

in which $E_k$ indicates an event that the $k$-th element will satisfy its limit state condition.

The reliability analysis by Monte Carlo simulation possesses a great advantage in that the system reliability of any kind of definition is easily introduced. For example, if the system limit state probability is represented by the probability that at least one of the $m$ crucial elements will satisfy its limit state condition, then

$$P_f = \text{Prob}(E_{k_1} \cup E_{k_2} \cup \cdots \cup E_{k_m}) \quad \cdots\cdots (12)$$

In this case, the estimator of the system limit state probability is given by

$$P_f = \frac{1}{N_s} \sum_{j=1}^{N_s} \delta(\boldsymbol{X}^{(j)}); \quad \delta(\boldsymbol{X}^{(j)}) = \begin{cases} 1 & \text{if } \sum_{l=1}^{m} \delta_{k_l}(X^{(j)}) \geqq 1 \\ 0 & \text{otherwise} \end{cases} \quad \cdots\cdots (13\cdot a, b)$$

The notation as $P_f^{MC}$ and $\delta^{MC}(\cdot)$ is used henceforth to indicate respectively the limit state probability and performance function evaluated by Monte Carlo simulation. This notation is used for both the element and system limit state probabilities and their associated performance functions, since the nature of the error in the limit state probability is common to both of them. $P_f^{MC}$ is an unbiased estimator of $P_f$, because one can easily show that

$$E[P_f^{MC}] = \frac{1}{N_s} E\left[\sum_{j=1}^{N_s} \delta^{MC}(\boldsymbol{X}^{(j)})\right] = E[\delta^{MC}(\boldsymbol{X})] = \bar{P}_f \quad \cdots\cdots (14)$$

in which $\bar{P}_f$ is the exact but unknown value of $P_f$. The error of the Monte Carlo solution can then be evaluated by the variance of $P_f^{MC}$ as

$$\text{Var}[P_f^{MC}] = \frac{1}{N_s^2} \text{Var}\left[\sum_{j=1}^{N_s} \delta^{MC}(\boldsymbol{X}^{(j)})\right] = \frac{1}{N_s} \text{Var}[\delta^{MC}(\boldsymbol{X})] = \frac{\bar{P}_f - \bar{P}_f^2}{N_s} \quad \cdots\cdots (15)$$

Thus, the relative error corresponding to $\pm l\sigma$ (standard deviation) confidence level is obtained as follows:

$$R_{error}^{(\pm l\sigma)} = l \frac{\sqrt{\text{Var}[P_f^{MC}]}}{\bar{P}_f} = l \sqrt{\frac{1-\bar{P}_f}{N_s \bar{P}_f}} \quad \cdots\cdots (16)$$

If the distribution function of $P_f^{MC}$ is Gaussian, the probabilities that the relative error in $P_f^{MC}$ is within the values given by Eq. (16) with $l=1$ and with $l=2$ are 68 % and 95 %, respectively.

For most structural systems, the limit state probability is small. Hence, the sample size $N_s$ must be large in order to obtain the estimator of $P_f$ with required accuracy. However, such $P_f$ is highly dependent on the tail shape of assumed probability distribution function where we usually have very scarce information. Thus, the allowable relative error may be taken as large as, for example, 1.0 because of the intrinsic uncertainty involved in $P_f$. Then, using the $\pm 1\,\sigma$ confidence level, the sample size may be determined as $N_s \approx 1/\bar{P}_f$. Obviously, the sample size must be taken much bigger than $1/\bar{P}_f$ if more accurate estimation is required. It is noted that the total number of random variables $N$ does not influence the error in $P_f^{MC}$ as shown above. Thus, stable $P_f^{MC}$ values are equally obtained regardless of the number of random variables involved. This fact provides a considerable advantage for the safety analysis based on the stochastic vector simulation.

## 4. MONTE CARLO INTEGRATION

If the limit state equation for an element or a system is given by $g(\boldsymbol{X}) \leqq 0$, then the limit state probability can also be evaluated directly from its definition as

$$P_f = \int_{D_X} f_X(\boldsymbol{t})\, d\boldsymbol{t} \quad \cdots\cdots (17)$$

in which $D_X$ indicates the domain in the $N$-dimensional space where $g(\boldsymbol{X}) \leqq 0$ and $f_X(\boldsymbol{t})$ is the joint probability density function of $\boldsymbol{X}$ as

$$f_X(\boldsymbol{t})=f_{X_1X_2\cdots X_N}(t_1, t_2, \cdots, t_N) \quad (18)$$

It is not practical to carry out the integration in Eq. (17) either analytically or numerically, because the components of $\boldsymbol{X}$ are generally correlated each other and the domain $D_X$ is very often irregular. Utilizing the modal decomposition, $\boldsymbol{X}$ is transformed into a stochastic vector $\boldsymbol{Z}$ whose components are uncorrelated each other. The $i$-th component $Z_i$ of $\boldsymbol{Z}$ is mean zero and has the variance $\lambda_i$. Then, $Z$'s are further standardized to have unit standard deviation by $z_i=Z_i/\sqrt{\lambda_i}$. Assume the standardized stochastic vector $\boldsymbol{z}=[z_1z_2\cdots z_N]^T$ to be Gaussian, its joint probability density function $f_z(\boldsymbol{t})$ is given by

$$f_z(\boldsymbol{t})=\frac{1}{(\sqrt{2\pi})^N}\exp\left(-\frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}\right) \quad (19)$$

Then, the limit state probability is evaluated by

$$P_f=\int_{D_Z} f_z(\boldsymbol{t})\,d\boldsymbol{t} \quad (20)$$

in which $D_z$ indicates the transformed domain for the $N$-fold integration. One can handle only Gaussian stochastic vectors by this approach, since the analytical form of the multi-dimensional joint probability density function can be constructed only for Gaussian cases.

Even in these cases, the analytical solution of Eq. (20) is still practically impossible to obtain because the domain $D_z$ is irregular. Hence, one looks for numerical integration methods. The straightforward approach to this problem is to adopt a multi-dimensional version of the trapezoidal or Simpson rule for numerical integration. However, such direct numerical integration methods are overly time-consuming if $N$ is large, and that is often the case for finite element systems with stochastic material properties. Thus, sample-mean (or crude) Monte Carlo method[15] is considered as a method of multi-dimensional numerical integration[3]. Then, $P_f$ is estimated by

$$P_f^I=\frac{A_N}{N_S}\sum_{j=1}^{N_S} f_z^*(\boldsymbol{t}^{(j)}); \quad f_z^*(\boldsymbol{t}^{(j)})=\begin{cases} f_z(\boldsymbol{t}^{(j)}) & \text{if } g(\boldsymbol{t}^{(j)})\leqq 0 \\ 0 & \text{otherwise}\end{cases} \quad (21\cdot\text{a, b})$$

in which $\boldsymbol{t}^{(j)}$ is the $j$-th realization of $\boldsymbol{z}$ generated by means of a computer simulation of the $N$ independent, uniformly distributed random variables over a hyper-area $A_N$. Since $\boldsymbol{z}$ is Gaussian, the domain of integration must be the entire domain $\Omega$ in $N$-dimension. However, if the probability density $f_z^*(\boldsymbol{t})$ outside $\pm d_i/2$ is negligibly small, the domain may be approximated by an appropriate truncated area, $\Omega_N$, as

$$\Omega_N(t_1, t_2, \cdots, t_N)=\Omega\left(-\frac{d_1}{2}\leqq t_1\leqq\frac{d_1}{2}\right)\cap\Omega\left(-\frac{d_2}{2}\leqq t_2\leqq\frac{d_2}{2}\right)\cap\cdots\cap\Omega\left(-\frac{d_N}{2}\leqq t_N\leqq\frac{d_N}{2}\right) \quad (22)$$

in which $\pm d_i/2$ indicates the range of sampling for $t_i$. Thus, $A_N$ is taken as $A_N=d_1d_2\cdots d_N$.

The estimator $P_f^I$ may be biased since it always contains the truncation error to some extent. The accuracy of the limit state probability estimated by Eq. (21) may depend on the dimension $N$ as well as the selection of the area $A_N$. If the variances, $\lambda_i$'s, are small enough for $i>M$, $N$ can be reduced to $M$ and consequently, $\Omega_N$ to $\Omega_M$ as

$$\Omega_M(t_1, t_2, \cdots, t_M)=\Omega\left(-\frac{d_1}{2}\leqq t_1\leqq\frac{d_1}{2}\right)\cap\Omega\left(-\frac{d_2}{2}\leqq t_2\leqq\frac{d_2}{2}\right)\cap\cdots\cap\Omega\left(-\frac{d_M}{2}\leqq t_M\leqq\frac{d_M}{2}\right) \quad (23)$$

and $A_N$ to $A_M$ as $A_M=d_1d_2\cdots d_M$. After this reduction of dimension, the error in $P_f^I$ is evaluated by its variance

$$\mathrm{Var}[P_f^I]=\left(\frac{A_M}{N_S}\right)^2\mathrm{Var}\left[\sum_{j=1}^{N_S} f_z^*(\boldsymbol{t}^{(j)})\right]=\frac{A_M^2}{N_S}\mathrm{Var}[f_z^*(\boldsymbol{t})] \quad (24)$$

Since $\boldsymbol{t}=[t_1t_2\cdots t_M]^T$ consists of uniformly distributed independent random variables, the probability density of $\boldsymbol{t}$ is

$$p(\boldsymbol{t})=\begin{cases} 1/A_M & \text{if } \boldsymbol{t} \text{ is inside } \Omega_M \\ 0 & \text{otherwise}\end{cases} \quad (25)$$

Then,

$$E[f_Z^*(t)]\simeq\int_{\Omega_M} f_Z^*(t)\,p(t)\,dt\simeq\frac{\bar{P}_f}{A_M};\quad E[(f_Z^*(t))^2]\simeq\int_{\Omega_M}(f_Z^*(t))^2 p(t)\,dt\simeq\frac{1}{A_M}\int_{\Omega_M}(f_Z^*(t))^2\,dt \quad \cdots\cdots (26\cdot a,\ b)$$

with $\bar{P}_f$=the exact value of $P_f$ as mentioned earlier. Introducing Eq. (26) into Eq. (24), one gets

$$\mathrm{Var}[P_f']\simeq\frac{1}{N_S}\left\{A_M\int_{\Omega_M}(f_Z^*(t))^2 dt-\bar{P}_f^2\right\} \quad \cdots\cdots (27)$$

Eq. (27) has a similar form with Eq. (15) except for the first component of the right hand side term. However, Var $[P_f']$ increases rapidly as the dimension $M$ becomes large because of $A_M$ involved.

In order to examine the relationship between Var $[P_f']$ and $M$, a simple limit state condition, which is a function of only one random variable, is considered as an example :

$$g(t)=g(t_k)=t_k-t_L \quad \cdots\cdots (28)$$

with $t_L$=the critical value for $t_k$. Assuming the range of the random sampling as $d_i=d$ for $i=1, 2, \cdots, M$, one can derive

$$A_M\int_{\Omega_M}(f_Z^*(t))^2 dt=d^M\int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{-\frac{d}{2}}^{\frac{d}{2}}\cdots\int_{-\frac{d}{2}}^{t_L}(f_Z(t))^2 dt_k dt_1\cdots dt_M=\frac{1}{2}\left(1+\frac{\mathrm{erf}(t_L)}{\mathrm{erf}(d/2)}\right)\left(\frac{d\cdot\mathrm{erf}(d/2)}{2\sqrt{\pi}}\right)^M \cdots (29)$$

in which erf $(\cdot)$ indicates the error function defined by

$$\mathrm{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^y \exp(-t^2)\,dt \quad \cdots\cdots (30)$$

If $d$ is taken as 6 representing $\pm 3\sigma$ range, $1.69^M$ is involved in Eq. (29). Also, if $d$ is taken as 8 representing $\pm 4\sigma$ range, $2.26^M$ is involved in Eq. (29). Hence, it is clearly observed that Var $[P_f']$ increases rapidly as $M$ and/or $d$ become large. However, we should take $d$ larger than about 6. Otherwise, the probability density $f_Z^*(t)$ outside $\pm d/2$ is not negligibly small. Unless these rules are observed, Monte Carlo integration method may result in substantially erroneous conclusions for the problems with a large number of random variables.

It is noted that various variance reduction techniques[15),16)] and their application[17)] are not available for the problems considered herein. For example, "importance sampling" is difficult to apply, since the limit state surface is highly irregular for the most cases. Also, "antithetic variates" and "stratified sampling" may require the huge sample size, since there are an enormous number of possible combinations in values of random variables. Furthermore, the response surface method[18)] is not efficient either for the same reason. Thus, the straightforward Monte Carlo integration may still be the best way among direct evaluation methods of $P_f$ from the joint density function.

## 5. NUMERICAL EXAMPLE

### (1) Uni-variate stochastic field

A computer program was developed by which we could evaluate the reliability of finite element systems involving stochastic fields. The finite element formulation for linear static problems is performed using a plain strain rectangular element. This program estimates the limit state probability of various kind, such as for individual elements, for failure modes and for the total system, by means of Monte Carlo technique. The two different Monte Carlo schemes, Monte Carlo simulation and Monte Carlo integration, are employed for evaluating such limit state probabilities.

As a numerical example, a diluvial gravelly ground model, which is recently being studied as a new possible location for nuclear facilities in Japan, is considered. The ground model is discretized into two hundred (200) finite elements as shown in Fig. 1. A fixed boundary condition is applied along the lower edge and horizontal displacements are constrained along the side edges of the model. A uniformly distributed deterministic vertical load which represents the weight of a reactor building is applied on the ground surface. The assumed medians of soil properties are angle of internal friction $\phi_0=35^\circ$ and Poisson's ratio $\nu_0=0.3$. The weight density and Young's modulus are assumed to be constants : $\gamma_t=2.0$ t/m$^3$ (19.6

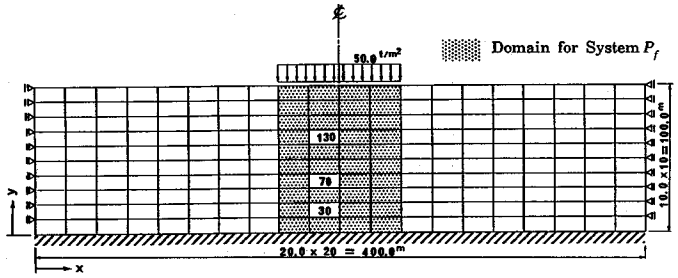



Fig. 1 Finite Element Model.

kN/m³) and $E=30\,000$ t/m² (294 000 kN/m²), respectively, since these parameters are less influential on the limit state probabilities. Mohr-Coulomb criterion as shown in Fig. 2 is introduced in order to construct the local (element) limit state equation for the $k$-th element as

$$g_k(X)=\left(\frac{\sigma_1+\sigma_3}{2}\right)_k \sin\phi_k-\left(\frac{\sigma_1-\sigma_3}{2}\right)_k \quad \cdots\cdots (31)$$

in which $\sigma_1$ and $\sigma_3$ indicate the maximum and minimum principal stresses with positive values in compression.

Firstly, $\sin\phi$ is assumed to be log-normal as

$$\sin\phi_k=\sin\phi_0 \exp(X_k\zeta_\phi) \quad \cdots\cdots (32)$$

in which $X_k$ is the Gaussian random variable with zero mean and unit standard deviation. Then, $X$'s compose a correlated stochastic vector $X=[X_1X_2\cdots X_n]^T$ with $n=200$. The logarithmic standard deviation $\zeta_\phi$ of $\sin\phi$ is assumed to be 0.1 just for demonstration. All other material properties are assumed to be deterministic in this case, then the stresses are also deterministic. Hence, the limit state equation of each element contains only one Gaussian random variable. Consequently, the exact solution for the local limit state probability exists:

$$P_{fk}=\Phi^{-1}\left(\frac{\ln S_k-\ln(\sin\phi_0)}{\zeta_\phi}\right) \text{ with } S_k=\left(\frac{\sigma_1-\sigma_3}{\sigma_1+\sigma_3}\right)_k \quad \cdots\cdots (33)$$

in which $\Phi^{-1}(\cdot)$ indicates the inverse standardized Gaussian distribution function. Thus, the accuracy of Monte Carlo methods can be examined by comparing their results with the exact solution obtained by Eq. (33).

The spatial correlation in soil parameters may be considered strong in the horizontal ($x$) direction and rather weak in the vertical ($y$) direction. Hence, the anisotropic auto-correlation function given in Eq. (3) is considered with an assumption that $d_x$ is four times longer than $d_y$. The shape of this auto-correlation function is depicted in Fig. 3. Numerical examples are provided for the three cases of $d_x$ and $d_y$, 320 and 80 m; 160 and 40 m; 80 and 20 m, in order to examine the effects of the correlation

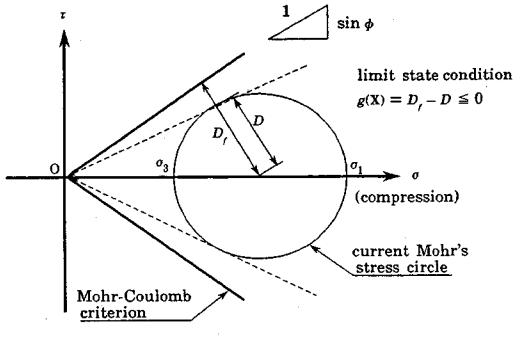



Fig. 2 Local (Element) Limit State Condition.

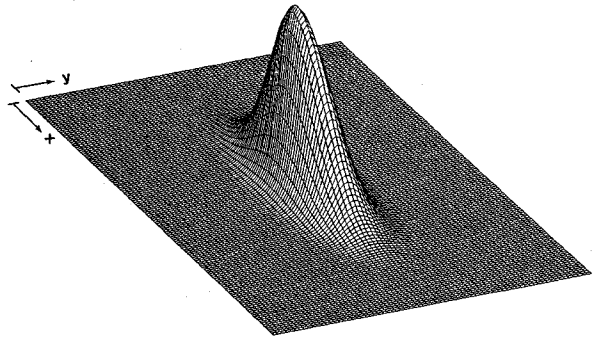



Fig. 3 Anisotropic Auto-Correlation Function.

distances on the accuracy of analysis. The result of eigenvalue analysis for the covariance matrix of $\boldsymbol{X}$ is depicted in Fig. 4. It is clearly observed that the number of significant modes (or effective independent random variables) is much less than the total number of modes (or correlated random variables) and that this tendency is more prominent when the correlation distances, $d_x$ and $d_y$, are large. The truncation mode number $M$ is determined such that the normalized sum of eigenvalues $P_\lambda(M)$ to be more than 95 % for Monte Carlo integration and more than 99.99 % for Monte Carlo simulation. If we take a larger $M$ for Monte Carlo integration, its accuracy gets worse due the increase of dimension in the integration. While for Monte Carlo simulation, the result does not change for a larger $M$.

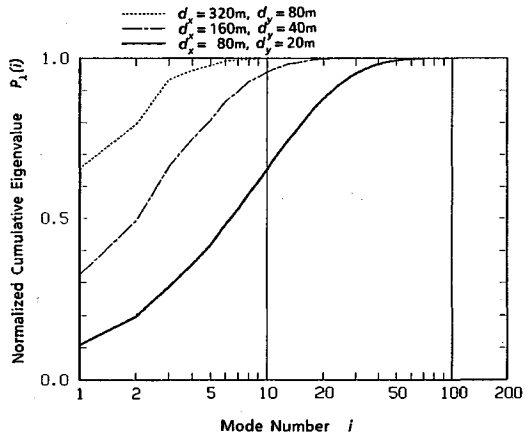
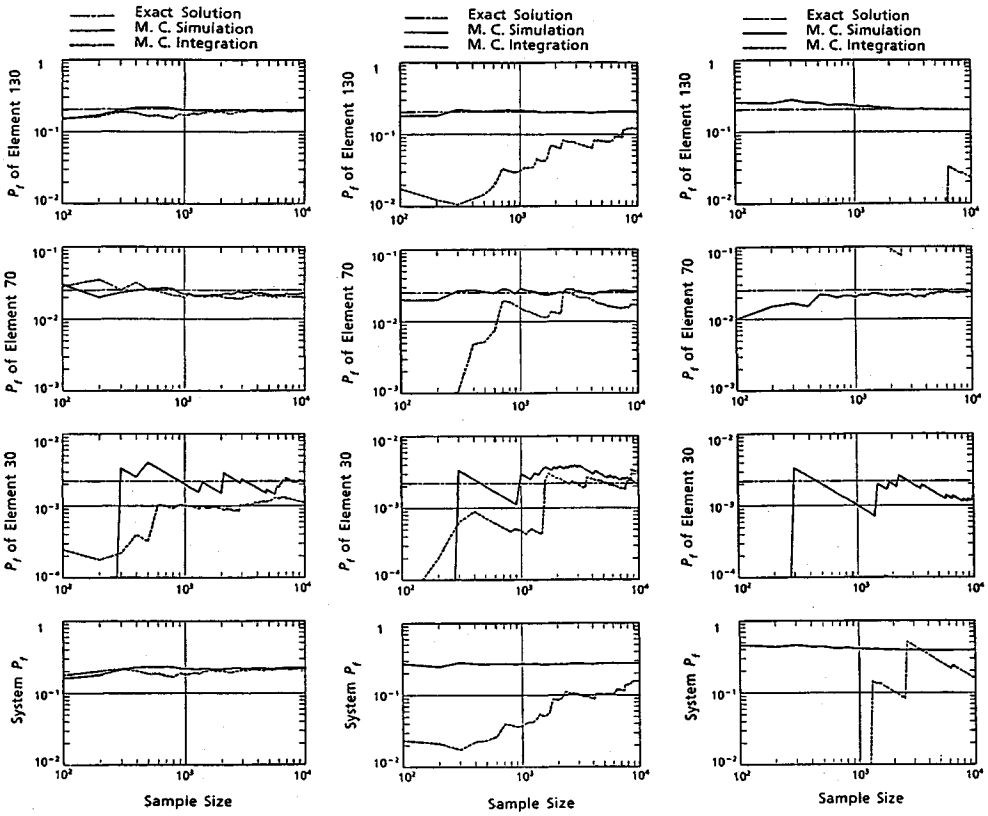



Fig. 4 Relationship between Mode Number and Normalized Sum of Eigenvalue.

The reliability analysis is carried out by those two Monte Carlo methods for three cases of the correlation distances. The relationship between the sample size and limit state probability of typical elements is plotted in Fig. 5. It is confirmed that the accuracy of Monte Carlo simulation is not affected by the correlation distances (or the number of independent random variables) and only a function of its sample size. On the other hand,




(a) $d_x$=320m, $d_y$=80m (b) $d_x$=160m, $d_y$=40m (c) $d_x$=80m, $d_y$=20m

Fig. 5 Relationship between Sample Size and Local and System Limit State Probabilities for Variation of $\sin\phi$.

the accuracy of Monte Carlo integration decreases drastically as the correlation distances decrease (or the number of independent random variables increases) and it is comparable with that of Monte Carlo simulation only when the correlation distances are very long. Note that, in Fig. 5 (c), the results by Monte Carlo integration are so erroneous that they are out of range of the plotting frame.

The system limit state probability is assumed here as the probability in which at least one of the forty elements (see Fig. 1) below the applied load satisfy their local limit state conditions. Such system $P_f$'s are also shown in Fig. 5 although they do not have exact solutions. The system $P_f$ becomes large as the correlation distances become small. The system $P_f$ by Monte Carlo simulation is also proved to be much more stable than that by Monte Carlo integration.

(2) Bi-variate stochastic field

As the second example, Poisson's ratio $\nu$ and strength parameter $\sin\phi$ are assumed to constitute a bi-variate stochastic field. Then Poisson's ratio for the $k$-th element is given by

$$\nu_k = \nu_0 \exp(X_{k+n}\zeta_\nu) \quad \cdots\cdots (34)$$

in which $X_{k+n}$ is the standardized Gaussian random variable with mean zero and unit standard deviation. Considering both Eqs. (32) and (34), $X$'s compose a correlated stochastic vector with dimension $N=400$. The logarithmic standard deviations, $\zeta_\phi$ and $\zeta_\nu$, are both assumed to be 0.1 and all the other parameters are deterministic.

The auto- and cross-correlation functions for the bi-variate two-dimensional stochastic field are given in Eq. (3). Two cases of cross-correlation between $\sin\phi$ and $\nu$, $\beta=0.0$ (independent) and $\beta=-0.8$ (strong negative correlation), are considered to demonstrate the effect of the cross-correlation. The negative correlation between $\sin\phi$ and $\nu$ has some rationality by way of the following two relationships; For linear elasticity problems, the lateral stress ratio $K_0$ for one-dimensional strain condition is represented by $K_0=\nu/(1-\nu)$. Also, there is a well-known empirical formula for granular soils by Jaky : $K_0=1-\sin\phi$. Although these two equations have been derived as the mean

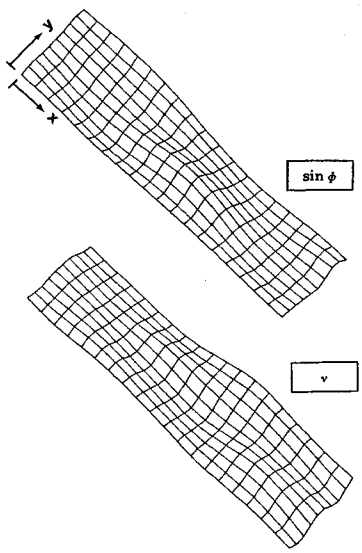



Fig. 6 Sample Bi-Variate Stochastic Field by Monte Carlo Simulation (Variation of $\sin\phi$ and $\nu$, $d_x=80$ m, $d_y=20$ m, $\beta=-0.8$).

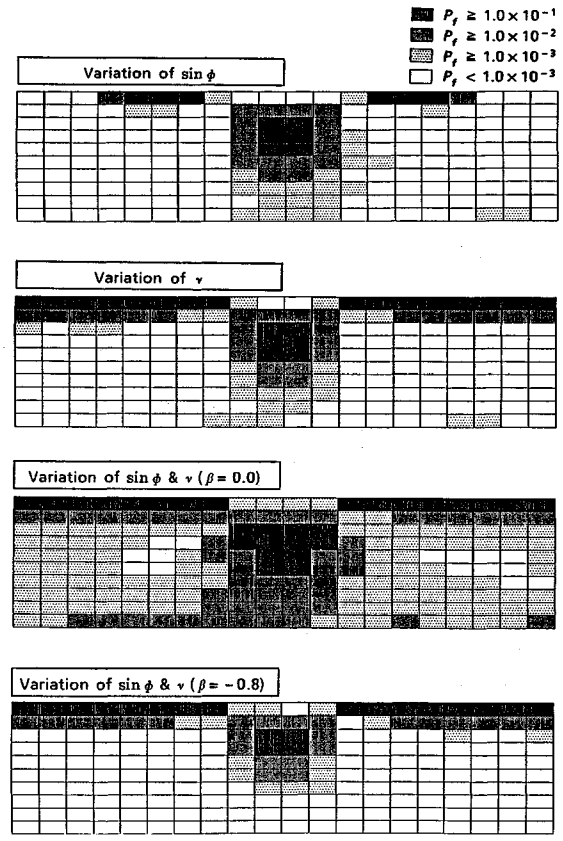



Fig. 7 Local Limit State Probability by Monte Carlo Simulation ($N_s=1\,000$, $d_x=80$ m, $d_y=20$ m).

relationship for deterministic materials, they still suggest some negative correlation between the deviatoric components in $\sin\phi$ and $\nu$.

A sample bi-variate stochastic field for $\beta=-0.8$ is depicted in Fig. 6 where the negative correlation is clearly observed. The reliability analysis is carried out by Monte Carlo simulation for the correlation distances, $d_x=80$ m and $d_y=20$ m, with the sample size 1 000. The local (element) limit state probabilities for the four cases of material property variability : variation of $\sin\phi$ only, $\nu$ only, $\sin\phi$ and $\nu$ ($\beta=0.0$), $\sin\phi$ and $\nu$ ($\beta=-0.8$), are shown in Fig. 7. Also, the relationship between the sample size and system $P_f$ is plotted in Fig. 8. It is noted that the value of $\beta$ has a significant influence on the local and system $P_f$'s. The negative cross-correlation between $\sin\phi$ and $\nu$ is found to be favorable to maintain the reliability defined by Mohr-Coulomb criterion. These results indicate that the correlational characteristics of material properties must be introduced properly when evaluating the structural safety.

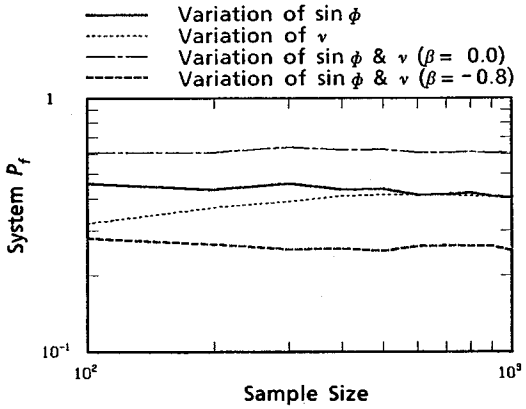



Fig. 8 Relationship between Sample Size and System Limit State Probability by Monte Carlo Simulation ($d_x=80$ m, $d_y=20$ m).

## 6. CONCLUSION

Structural safety analysis methods based on Monte Carlo techniques are presented. When material properties in structural systems exhibit spatial variabilities, the finite element method is the most convenient solution approach to evaluate the response and resulting reliability of such systems. The reliability analysis of such finite element systems is by no means easy because of, primarily, a large number of random variables involved and the complexity of limit state surfaces. Since the first-order second-moment methods bear limitation in such cases, Monte Carlo methods are highlighted as a powerful alternative.

The accuracy of two different Monte Carlo methods are investigated. "Monte Carlo simulation" implies a technique for generating correlated random variables representing underlying multi-dimensional multivariate stochastic fields. Then, introducing such sample random variables into the finite element system, the safety statement is made sample by sample. This Monte Carlo simulation is recommended here, because its accuracy does not depend on the number of random variables with the appropriate sample size. On the contrary, accuracy of "Monte Carlo integration", where a numerical integration of the joint density function of the random variables is carried out by means of sample-mean Monte Carlo method, goes down dramatically as the number of random variables becomes large.

Numerical examples indeed verified these assertions. The local and system reliabilities of a ground model under a vertical surface load are obtained accurately by Monte Carlo simulation. However, Monte Carlo integration can provide good results only when the number of random variables are very small. Also, the spatial correlation distances in the auto-correlation function and the degree of cross-correlation are demonstrated to have strong influences on the limit state probability.

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