Simulation of Stochastic Fields by Statistical Preconditioning

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Abstract: A new stochastic fields simulation technique that can realize prescribed means and covariances with a substantially smaller sample size than that of other existing methods is developed. The method utilizes the modal decomposition of the covariance matrix of the correlated random vector and the spectral representation of random processes. The decreasing feature of the eigenvalues of the covariance matrix and the orthogonality of the trigonometric functions are taken advantage of for generating sets of independent random variables. The generated discretized stochastic field is Gaussian by virtue of the central limit theorem. The sample functions of the discretized stochastic field precisely reproduces, when ensemble-averaged, the prescribed zero-mean and covariance function. Hence, the proposed statistical preconditioning technique will, in general, dramatically reduce the large computational effort that Monte Carlo simulation involving stochastic fields would otherwise entail.

Introduction

Monte Carlo simulation techniques are of considerable usefulness for solving stochastic problems in the general area of civil engineering and engineering mechanics because of their versatility and algorithmic straightforwardness. Examples of such solutions include the estimation of: (1) Eigenvalue and structural response variabilities arising from spatial material property and/or geometry variations within structural systems (problems of system stochasticity) as performed by Shinozuka and Astill (1972), Astill et al. (1972), and Yamazaki, et al. (1988); and (2) dynamic response characteristics of nonlinear structures under temporally and spatially random loading conditions as demonstrated by Shinozuka and his associates in a number of papers referenced in a summary by Shinozuka (1977). However, the degree of accuracy of the resulting ensemble statistics such as mean values and standard deviations is, in general, satisfactorily high only when sample size is sufficiently large. It is pointed out that the statistical fluctuation of sample means, sample correlation functions, and the like, that arise from generating a finite number of sample functions of the stochastic process or field involved, influence significantly the resulting response statistics. Hence, Monte Carlo simulation techniques usually require a substantial amount of computational effort.

Then, it is the purpose of this study to develop a method of generating sample functions of a discretized Gaussian stochastic field (Gaussian stochastic vector) in which the sample functions reproduces exactly the prescribed zero-mean and auto-correlation function. Such a method would make it possible to generate more reliable response statistics on a substantially

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smaller number of sample functions, and hence it represents a major improvement over currently available Monte Carlo methods for evaluating the response of engineering systems when the excitation consists of stochastic fields or random processes. Numerical examples for the response statistics will be shown in what follows.

In general, the simulation and generation of sample functions of stochastic fields can be performed by means of: (1) Spectral representation; (2) ARMA (auto-regressive moving average) modeling; and (3) covariance matrix decomposition procedures.

The spectral representation method enables one to simulate stochastic fields and generate their sample functions, resulting in highly accurate spatial statistics because of the periodicity and orthogonality of the trigonometric functions used for expansion. In principal, the extension of the spectral representation method from one-dimensional univariate cases to multidimensional, multivariate cases can be easily achieved as demonstrated by Shinozuka (1972) and Shinozuka and Jan (1972). The method has further been modified by Shinozuka (1987). Digital generation of their sample functions can be carried out efficiently with the aid of the Fast Fourier Transform (FFT) algorithm as applied first by Yang (1972) and Shinozuka (1974). Also, the spectral representation can be extended to nonhomogeneous fields by introducing the evolutionary spectral density concept developed by Priestley (1965). In a more recent paper, Shinozuka, et al. (1987) utilized the evolutionary power concept both in the time and space domain to model and digitally generate the ground surface motion due to a propagating seismic wave. Indeed, one of the purposes of the present study is to make it possible to perform such a digital simulation more accurately.

A large number of samples is usually required in order to obtain stable ensemble statistics within a finite domain of the space in which the sample functions are generated. The stability of the ensemble statistics of such sample functions is of great importance, particularly when they are used as an input to the finite element analyses. This is because the underlying ergodicity of the input stochastic field cannot be taken advantage of when the Monte Carlo solution calls for ensemble averaging on usually nonergodic outputs of the finite element analyses. In passing, it is noted that once homogeneous Gaussian stochastic fields are generated, their transformation into non-Gaussian homogeneous fields, while maintaining the prescribed correlational characteristics, is possible as demonstrated by Yamazaki and Shinozuka (1988).

The covariance matrix decomposition methods, such as by Cholesky decomposition and by modal (or spectral) decomposition, enable one to simulate stochastic fields and generate their sample functions at discretized points if their correlation functions are given. So long as the covariance matrix is known, homogeneity of underlying stochastic fields is not required for the purpose of sample function generation. Therefore, these methods also enable one to generate sample functions of nonhomogeneous multivariate and multidimensional stochastic fields as easily as for those of homogeneous univariate, one-dimensional stochastic fields. The algorithm of the decomposition methods is straightforward, and once the decomposition has been achieved, the sample functions can be generated with considerable ease. However, covariance matrix decomposition methods may also require a large number of samples until stable ensemble statistics can be obtained with high accuracy.

ARMA model representation of stochastic fields has recently received considerable attention from the viewpoint of simulating stochastic processes and fields and generating their sample functions, for example, as studied by Samaras, et al. (1985) and Naganuma, et al. (1987). The advantage of this representation lies in the fact that a substantially smaller memory space and CPU time are required for computer generation of sample functions because of the recursive form of the equations used to represent the stochastic processes and fields. From the experience of the second writer, however, the method usually requires a larger sample size in order to achieve the same level of statistical accuracy as the spectral representation method would with a smaller sample size. Thus a very long time series was generated in the paper by Samaras, et al. (1985) for demonstrating the statistical accuracy of ARMA model.

Considering these features of simulation and generation of stochastic processes and fields, a hybrid method consisting of covariance matrix decomposition and spectral representation is developed. Here, the decreasing characteristics of eigenvalues of the covariance matrices and orthogonality of trigonometric functions are taken advantage of. Numerical examples demonstrate that the sample generated by the proposed method precisely reproduces the target mean values as well as the correlation function and also exhibits Gaussian characteristics very well.

Modal Decomposition Method

As mentioned above, discretized Gaussian homogeneous stochastic fields can be simulated and their sample functions generated with the aid of the methods based on covariance matrix decomposition. These methods make it possible to transform a set of independent Gaussian random variables into a set of correlated Gaussian random variables with a prescribed covariance matrix. Two methods are well known and frequently used; one method utilizes Cholesky decomposition and the other modal decomposition. In the present study, the modal decomposition method [see, e.g., Ditlevsen (1981), Liu et al. (1987)] is used primarily for the reason that it has certain features useful for later analytical development.

In order to take advantage of the method, the stochastic field must be discretized first into a number, say n, of subdomains. These subdomains must be small enough so that, for example, over each subdomain, the value of the stochastic field may be considered to be constant, at least in approximation. Hence, the stochastic field is represented by a set of n random variables, which are in general correlated. Obviously, such a discretization is perfectly consistent with the finite element formulation of structural problems where, for example, the spatial randomness of a material property or a geometry parameter is modeled as a stochastic field and the structure must be divided into n finite elements, each element having the value of the stochastic field at its representative location (e.g., at the centroid) as the representation of the material property for the element. Note that there is another way to determine this representative value for the element: taking local average of the variation within the element [e.g., Vanmarcke (1983), Der Kiureghian and Ke (1988)]. In either way, the stochastic field for the structure is approximated by its discretized version $\mathbf{X} = [X_1 X_2 \ldots X_n]^T$. We emphasize at this point that the purpose of this study is not to examine the

adequacy of particular methods of finite element discretization, but rather to develop a method of simulating a stochastic field thus discretized with perfect reproducibility of specified mean value and autocorrelation function. The subject of finite element discretization is considered as one of the most logical applications of the proposed method.

The following comments are in order with respect to the discretization of the stochastic field mentioned above: (1) Even if there are two or more material property parameters, they can also be discretized by means of the covariance decomposition method, to be described next, as long as they are characterized as homogeneous or even nonhomogeneous multivariate and multidimensional stochastic fields; and (2) for obvious reasons, the size of the finite elements must also satisfy the usual requirements from the viewpoint of stress and strain gradients.

The covariance matrix of $\mathbf{X}$ is defined by

$$C_{XX} = E\{[\mathbf{X} - E(\mathbf{X})][\mathbf{X} - E(\mathbf{X})]^T\} \quad (1a)$$

$$C_{XX} = \begin{bmatrix} \text{var } (X_1) & \text{cov } (X_1,X_2) & \dots & \text{cov } (X_1,X_n) \\ \text{cov } (X_2,X_1) & \text{var } (X_2) & \dots & \text{cov } (X_2,X_n) \\ \vdots & \vdots & \dots & \vdots \\ \text{cov } (X_n,X_1) & \text{cov } (X_1,X_2) & \dots & \text{var } (X_n) \end{bmatrix} \quad (1b)$$

Assuming homogeneity and zero mean of the stochastic field, ij-component of C_{XX} is obtained from the auto-correlation function $R_{XX}(\cdot)$ of the stochastic field as

$$C_{XXij} = \text{cov } (X_i,X_j) = R_{XX}(\xi_{ij}) \quad (2)$$

in which ξ_{ij} is a separation vector between two points i and j.

The eigenvalues and eigenvectors of C_{XX} can be obtained by solving the following eigenequation:

$$C_{XX}\Phi_X = \Phi_X\Lambda_X \quad (3)$$

where Λ_X is the (diagonal) eigenvalue matrix and Φ_X is the modal matrix as follows:

$$\Lambda_X = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}; \qquad \Phi_X = [\phi_1\phi_2 \dots \phi_n] \quad (4)$$

where λ_i is the ith eigenvalue of C_{XX} such that

$$\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_n \quad (5)$$

Also ϕ_i is the ith eigenvector of C_{XX} normalized to have a Euclidean length of 1, and orthogonal with ϕ_j $(i \neq j)$. Therefore

$$\Phi_X^T\Phi_X = \mathbf{I} \quad (6)$$

in which $\mathbf{I}$ is an identity matrix. Since the covariance matrix $\mathbf{C}_{XX}$ is symmetric, all the eigenvalues and eigenvectors are obtained as real values. If a vector $\mathbf{Z} = (Z_1Z_2 \dots Z_n)^T$ is introduced as

$$\mathbf{Z} = \Phi_X^T\mathbf{X} \quad (7)$$

then, the expected value of $\mathbf{Z}$ is

$$E(\mathbf{Z}) = \Phi_X^T E(\mathbf{X}) \quad (8)$$

Eq. 8 implies that if $\mathbf{X}$ is mean zero, $\mathbf{Z}$ is also mean zero. Utilizing Eqs. 1, 3, and 7, the covariance matrix of $\mathbf{Z}$ can be shown to be

$$C_{ZZ} = E\{[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T\} = \Lambda_X \quad (9)$$

Since $\mathbf{C}_{ZZ}$ is found to be a diagonal matrix, Z_i $(i = 1, 2, \dots, n)$ are uncorrelated, if not independent, and their variances are λ_i $(i = 1, 2, \dots, n)$, respectively. Thus, by generating a set of independent random variables $\mathbf{Z}$, a corresponding set of correlated random variables $\mathbf{X}$ can be easily obtained as

$$\mathbf{X} = (\Phi_X^T)^{-1}\mathbf{Z} = \Phi_X\mathbf{Z} \quad (10)$$

In general, the modal decomposition method appears to have wider applicability compared with the Cholesky decomposition method, due primarily to the fact that: (1) It works well even when the components of $\mathbf{X}$ are highly correlated each other; and (2) the number of independent random variables Z_i to be generated in Eq. 10 may be reduced from n to M $(\leqq n)$ as will be seen later.

The modal decomposition method can also be applied to a multivariate stochastic fields problem. Consider an m-variate random vector $\mathbf{X}$ with dimension $n \times m$ as

$$\mathbf{X} = \{[\mathbf{X}^{(1)}]^T[\mathbf{X}^{(2)}]^T \dots [\mathbf{X}^{(m)}]^T\}^T \quad (11)$$

with

$$\mathbf{X}^{(i)} = [X_1^{(i)}X_2^{(i)} \dots X_n^{(i)}]^T \qquad (i = 1, 2, \dots, m) \quad (12)$$

The covariance matrix of the random vector $\mathbf{X}$ is represented by

$$C_{XX} = \begin{bmatrix} C_{XX}^{11} & C_{XX}^{12} & \dots & C_{XX}^{1m} \\ C_{XX}^{21} & C_{XX}^{22} & \dots & C_{XX}^{2m} \\ \cdot & \cdot & \dots & \cdot \\ C_{XX}^{m1} & C_{XX}^{m2} & \dots & C_{XX}^{mm} \end{bmatrix} \quad (13)$$

and each submatrix represents

$$C_{XX}^{ij} = \begin{bmatrix} \text{cov } [X_1^{(i)},X_1^{(j)}] & \text{cov } [X_1^{(i)}X_2^{(j)}] & \dots & \text{cov } [X_1^{(i)},X_n^{(j)}] \\ \text{cov } [X_2^{(i)},X_1^{(j)}] & \text{cov } [X_2^{(i)},X_2^{(j)}] & \dots & \text{cov } [X_2^{(i)},X_n^{(j)}] \\ \vdots & \vdots & \dots & \vdots \\ \text{cov } [X_n^{(i)},X_1^{(j)}] & \text{cov } [X_n^{(i)},X_2^{(j)}] & \dots & \text{cov } [X_n^{(i)},X_n^{(j)}] \end{bmatrix} \quad (14)$$

From a cross-correlation function $R_{XX}^{ij}(\cdot)$ between the zero-mean random vectors $\mathbf{X}^{(i)}$ and $\mathbf{X}^{(j)}$, the kl-component of the above submatrix is obtained as

$$\text{cov } [X_k^{(i)},X_l^{(j)}] = R_{XX}^{ij}(\xi_{kl}) \quad (15)$$

where ξ_{kl} is a separation vector between two points k and l with the following characteristics:

$$\xi_{kl} = -\xi_{lk} \quad (16)$$

Therefore the submatrices $\mathbf{C}_{XX}^{ij}$ $(i, j = 1, 2, \ldots, m)$ are generally not symmetric except when $R_{XX}^{ij}(\xi_{kl})$ is an even function of ξ_{kl}. There is a general relationship between a pair of cross-correlation functions such as

$$R_{XX}^{ij}(\xi_{kl}) = R_{XX}^{ji}(-\xi_{kl}) \quad (17)$$

From Eqs. 16 and 17, one obtains

$$R_{XX}^{ij}(\xi_{kl}) = R_{XX}^{ji}(\xi_{lk}) \quad (18)$$

Eq. 18 implies that the covariance matrix $\mathbf{C}_{XX}$ is always symmetric, regardless of the form assumed for the auto- and cross-correlations. Thus the eigenvalues and eigenvectors of C_{XX} can be obtained as real numbers. The rest of the procedure for the multivariate case is the same as that for the univariate case except that the total dimensions of the vectors $\mathbf{X}$ and $\mathbf{Z}$ become $n \times m$ instead of n.

Statistical Preconditioning

In Eq. 10, each component Z_i of the vector $\mathbf{Z}$ has mean zero and variance λ_i. The standard method to generate Z's is to take advantage of a well-tested computer program that is supposed to generate independent standardized Gaussian random numbers [e.g., GGNML in "The IMSL libraries" (1986)]. However, common experience indicates that a fairly large sample size is necessary for the simulated covariance matrix $\mathbf{C}_{ZZ}^{*}$ of $\mathbf{Z}$ to approach the target matrix $\mathbf{C}_{ZZ}$. Especially, the off-diagonal terms of C_{ZZ}^{*} require a large sample size before they approach zero. Here, the sample covariance matrix C_{ZZ}^{*} is obtained by

$$C_{ZZ}^{*} = \frac{1}{N_s}\sum_{j=1}^{N_s}\{Z(j)[Z(j)]^T - (\bar{Z}^*)(\bar{Z}^*)^T\} \quad (19)$$

and

$$\bar{Z}^* = \frac{1}{N_s}\sum_{j=1}^{N_s} Z(j) \quad (20)$$

where N_s is the sample size and $\mathbf{Z}(j)$ is the jth sample vector of $\mathbf{Z}$, and $\bar{\mathbf{Z}}^*$ is the sample mean vector of $\mathbf{Z}$.

If sample vector $\mathbf{Z}(i)$ is independent of each other, the estimators evaluated by Eqs. 19 and 20 approach the assumed covariance matrix and mean vector when $N_s \to \infty$ by the law of large numbers. Note that, however, even when $\mathbf{Z}(j)$ is correlated with each other, Eqs. 19 and 20 are still utilized for evaluating sample mean values and sample covariances in several statistical approaches such as the point estimate method (Rosenblueth 1975, 1981) and the Latin hypercube sampling technique (Mckay et al. 1979; Bazănt and Liu 1985). In these approaches, the sample size N_s is a relatively small number, not a large number as in the standard Monte Carlo methods. Thus the law of large numbers is not utilized. A simulation method we will propose in this paper is based on the similar idea: by introducing a proper correlation between each sample trial, the sample size may be reduced. Indeed, such an idea is often used in the standard sampling techniques, so-called variance reduction techniques [e.g., Rubinstein (1981)].

In order to generate samples of a random vector $\mathbf{Z}$ that will exactly produce the target C_{ZZ} with a substantially small sample size, a trigonometric expansion of the following form is considered for the ith component of the vector $\mathbf{Z}(j)$:

$$Z_i(j) = \sqrt{2}C_i \sum_{m=1}^{N_f} \cos(\omega_k j\Delta t + \psi_k)$$

$$(i = 1, 2, \ldots, n) \qquad (j = 1, 2, \ldots, N_s) \quad (21)$$

with $k = (m-1)n + i$, $C_i = \sqrt{\lambda_i/N_f}$. N_f = number of cosines to be added, ψ_k = random phase angle uniformly distributed between 0 and 2π, and ω_k = the kth circular frequency.

Eq. 21 has the form of well-known spectral representation for one-dimensional stationary stochastic processes although in this case the j-axis indicates sample number, not the time step. Note that when generating $Z_i(j)$, the jth sample of the ith component of Z, the summation involves frequencies, ω_i, ω_{n+i}, ω_{2n+i}, $\ldots$, $\omega_{(Nf-1)n+i}$ at $t = j\Delta t$. If Δt, N_s, and ω_k are determined properly, Eq. 21 exactly satisfies the target mean value and correlational characteristics for Z_i, as will be demonstrated next. It is possible to choose $N_f = 1$ if the form of the distribution function of Z_i is not important. If, however, a large number of N_f is assumed, the sample distribution of Z_i will approach a Gaussian distribution by virtue of the central limit theorem. The fundamental period T of the sinusoids to be used for generation is $T = T_1 = N_s\Delta t$, as shown in Fig. 1. Then the kth frequency ω_k has a period T_k such that

$$\omega_k = \frac{2\pi}{T_k} = \frac{2\pi k}{T} \quad \text{or } T_k = \frac{T}{k} \qquad (k = 1, 2, \ldots, nN_f) \quad (22)$$

Hence, there are k cycles of the sinusoids with frequency ω_k within duration T. In order to exactly satisfy the first two target moments by the corresponding ensemble averages that represent the averaging with respect to j, the interval Δt is chosen to be one-quarter of the shortest period T_{nNf}:

$$\Delta t = \frac{T_{nN_f}}{4} = \frac{\pi}{(2\omega_{nN_f})} \quad (23)$$

Under these assumptions (Eqs. 21, 22, and 23), the sample mean value is

$$\bar{Z}_i^* = \frac{\sqrt{2}}{N_s} C_i \sum_{m=1}^{N_f}\sum_{j=1}^{N_s} \cos(\omega_k j\Delta t + \psi_k) \quad (24a)$$

$$\bar{Z}_i^* = \frac{\sqrt{2}}{N_s} C_i \sum_{m=1}^{N_f}\sum_{j=1}^{N_s/4} \left[\cos(\omega_k j\Delta t + \psi_k) + \cos\left(\omega_k j\Delta t + \psi_k + \frac{k}{2}\pi\right)\right.$$

$$\left. + \cos(\omega_k j\Delta t + \psi_k + k\pi) + \cos\left(\omega_k j\Delta t + \psi_k + \frac{3k}{2}\pi\right)\right] = 0 \quad (24b)$$

for integer k. Note that the existence of random phase angle ψ_k in each cosine function does not influence this conclusion. The sample variance C_{ZZii}^{*} of Z_i

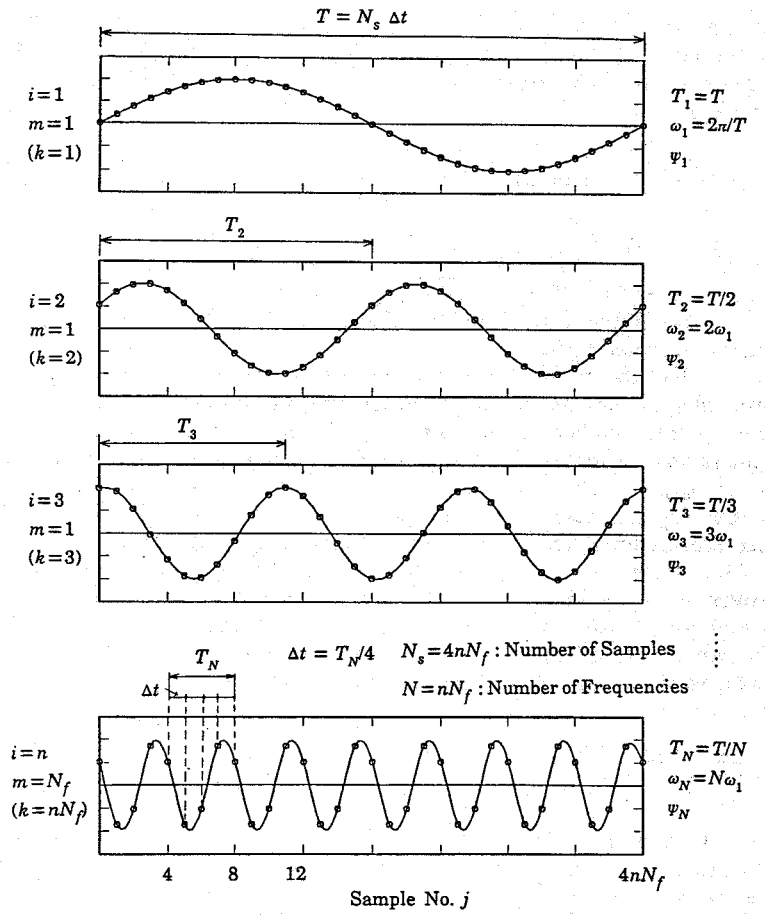

FIG. 1. Sinusoidal Functions Used for Generation of $Z_i(j)$

can be shown to be λ_i by the following algebraic manipulation

$$C^*_{ZZii} = \frac{1}{N_s}\sum_{j=1}^{N_s}[Z_i(j)]^2 = \frac{2}{N_s}C_i^2\sum_{j=1}^{N_s}\left[\sum_{m=1}^{N_f}\cos(\omega_k j\Delta t + \psi_k)\right]^2 \quad \text{(25a)}$$

$$C^*_{ZZii} = \frac{2}{N_s}\frac{\lambda_i}{N_f}\sum_{m=1}^{N_f}\sum_{j=1}^{N_s}\cos^2(\omega_k j\Delta t + \psi_k) = \lambda_i \quad \text{(25b)}$$

In deriving Eq. 25, the orthogonality of cosine functions are used:

$$\sum_{j=1}^{N_s}\cos(\omega_\alpha j\Delta t + \psi_\alpha)\cos(\omega_\beta j\Delta t + \psi_\beta) \quad \text{(26a)}$$

$$\sum_{j=1}^{N_s} = \sum_{j=1}^{N_s}\frac{1}{2}\{\cos[(\omega_\alpha - \omega_\beta)j\Delta t + \psi_\alpha - \psi_\beta] \quad \text{[Continued]}$$

$$+ \cos[(\omega_\alpha + \omega_\beta)j\Delta t + \psi_\alpha + \psi_\beta]\} \quad \text{(26b)}$$

$$\sum_{j=1}^{N_s} = 0 \quad \text{if } \omega_\alpha \neq \omega_\beta, \qquad \sum_{j=1}^{N_s} = \frac{N_s\cos(\psi_\alpha - \psi_\beta)}{2} \quad \text{if } \omega_\alpha = \omega_\beta \quad \text{(26c)}$$

Using different sets of N_f frequencies for different Z's, the sample covariance of Z_i and Z_l can be shown to be

$$C^*_{ZZil} = \frac{1}{N_s}\sum_{j=1}^{N_s}Z_i(j)Z_l(j) = 0 \quad \text{if } i \neq l \quad \text{(27)}$$

Here, the orthogonality of cosine functions is also utilized.

Eqs. 24, 25, and 27 show that the target expected value and covariance matrix of $\mathbf{Z}$ are exactly reproduced by the sample of size N_s by using Eq. 21, where the sample size N_s is

$$N_s = \frac{T}{\Delta t} = \frac{4T_1}{T_{nN_f}} = 4nN_f \quad \text{(28)}$$

It is noted that the number N_f of cosine functions to be summed to generate Z_i does not necessarily have to be identical for all values of i, although in this study N_f is assumed to be independent of i.

The N_s represented by Eq. 28 will be a large number if the dimension of the random vector n is large. However, Eq. 10 can be approximated in a truncated form by using only the first M modes:

$$X \approx [\phi_1\phi_2 \ldots \phi_M][Z_1Z_2 \ldots Z_M]^T \quad \text{(29)}$$

if a positive integer M is found such that $M < n$ and $\sqrt{\lambda_{M+1}} << \sqrt{\lambda_1}$. Use of Eq. 29 instead of Eq. 10 obviously reduces the number of Z's to be generated from n to M.

The orthogonality of the cosine functions is also valid if $\omega_\alpha = \omega_\beta$ and $\psi_\beta = \psi_\alpha + \pi/2$ in Eq. 26. Thus one can conveniently use the same frequency twice from the following condition, still maintaining the orthogonalities between Z_i and Z_l.

$$\omega_{k+M/2} = \omega_k \quad \text{and} \quad \psi_{k+M/2} = \psi_k + \frac{\pi}{2} \quad \text{(30)}$$

where k is such that $k = (m-1)M + i$ with $1 \leqq i \leqq M/2$ and $1 \leqq m \leqq N_f$ (see Table 1). Note that the sample mean (Eq. 24) remains zero after introducing the relationships in Eq. 30. Because of this, however, the sample size becomes

$$N_s = \frac{4T_1}{T_{N_fM/2}} = 2MN_f \quad (M\text{: even number}) \quad \text{(31)}$$

where N_f must be large if the simulated random vector $\mathbf{Z}$ is to be approximately Gaussian. Substituting Eq. 21 into Eq. 29

$$X_s(j) = \sqrt{2}\sum_{i=1}^{M}\sum_{m=1}^{N_f}\phi_{si}\sqrt{\frac{\lambda_i}{N_f}}\cos(\omega_k j\Delta t + \psi_k) \quad (s = 1, 2, \ldots, n) \quad \text{(32)}$$

in which ϕ_{si} is the sth component of vector ϕ_i. If $M \times N_f$ is large, $X_s(j)$ will

TABLE 1. Mode Number and Corresponding Frequencies and Random Phase Angles

Mode number (1)	Frequencies to superpose (2)	Random phase angles (3)
1	$\omega_1 \quad \omega_{M/2+1} \quad \dots \quad \omega_{(Nf-1)M/2+1}$	$\psi_1 \quad \psi_{M/2+1} \quad \dots \quad \psi_{(Nf-1)M/2+1}$
2	$\omega_2 \quad \omega_{M/2+2} \quad \dots \quad \omega_{(Nf-1)M/2+2}$	$\psi_2 \quad \psi_{M/2+2} \quad \dots \quad \psi_{(NF-1)M/2+2}$
⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮
$M/2$	$\omega_{M/2} \quad \omega_M \quad \dots \quad \omega_{NfM/2}$	$\psi_{M/2} \quad \psi_M \quad \dots \quad \psi_{NfM/2}$
$M/2 + 1$	$\omega_1 \quad \omega_{M/2+1} \quad \dots \quad \omega_{(Nf-1)M/2+1}$	$\psi_1 + \pi/2 \quad \psi_{M/2+1} + \pi/2 \quad \dots \quad \psi_{(Nf-1)M/2+1} + \pi/2$
$M/2 + 2$	$\omega_2 \quad \omega_{M/2+2} \quad \dots \quad \omega_{(Nf-1)M/2+2}$	$\psi_2 + \pi/2 \quad \psi_{M/2+2} + \pi/2 \quad \dots \quad \psi_{(Nf-1)M/2+2} + \pi/2$
⋮	⋮ ⋮ ⋮	⋮ ⋮ ⋮
M	$\omega_{M/2} \quad \omega_M \quad \dots \quad \omega_{NfM/2}$	$\psi_{M/2} + \pi/2 \quad \psi_M + \pi/2 \quad \dots \quad \psi_{NfM/2} + \pi/2$

have a Gaussian distribution asymptotically by virtue of the central limit theorem. Thus, even when $N_f = 1$, $X_s(j)$ is asymptotically Gaussian if M is sufficiently large. It is noted that the sample mean value is still exactly equal to zero while the sample covariances are approximately equal to the target values, even after mode truncation. Obviously, if the modes are not truncated, the sample covariances are identical to the target covariances.

Characteristics of Eigenvalues

The characteristics of eigenvalues and eigenvectors of the covariance matrix $\mathbf{C}_{XX}$ are investigated. If the stochastic field is homogeneous, the trace of $\mathbf{C}_{XX}$ represents the sum of the variances of X's, which is henceforth referred to as "total power" for convenience:

$$Tr(C_{XX}) = n\sigma^2 \quad \dots\dots (33)$$

in which σ is the standard deviation of the stochastic field. Also, there is the following relationship between the eigenvalues and the trace of a square matrix [e.g., Parlett (1980)]

$$Tr(C_{XX}) = \sum_{i=1}^{n} \lambda_i \quad \dots\dots (34)$$

Hence

$$\sum_{i=1}^{n} \lambda_i = n\sigma^2 \quad \dots\dots (35)$$

There are two limiting cases for $\mathbf{C}_{XX}$: The first is the case where $X_i = X_j$ for all values of i and j. This implies that

$$C_{XX} = \sigma^2 \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \quad \dots\dots (36)$$

In this case, the eigenvalues result in

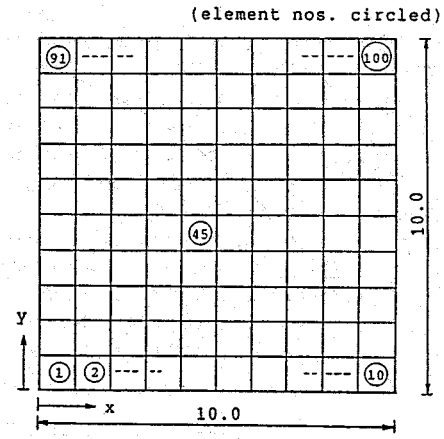

FIG. 2. Finite Element Discretization

$$\lambda_i = n\sigma^2 \quad (i = 1) \qquad \lambda_i = 0 \quad (i = 2, 3, \dots, n) \quad \dots\dots (37)$$

and the normalized eigenvector of the first mode is

$$\phi_1 = \left(\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}}\right)^T \quad \dots\dots (38)$$

Eq. 37 indicates that only the first eigenvalue is meaningful, and hence the stochastic field is reduced to one random variable. Since only one eigenvalue is superposed ($M = 1$) for this limiting case, N_f must be large for the X's to be asymptotically Gaussian.

The second limiting case involves a covariance matrix of the following form:

$$\mathbf{C}_{XX} = \sigma^2 \mathbf{I} \quad \dots\dots (39)$$

in which $\mathbf{I}$ is an identity matrix of appropriate dimension. This represents a discretized version of a two-dimensional white noise. The eigenvalue matrix of Eq. 39 is known as $\Lambda_X = \sigma^2 \mathbf{I}$. In this case, the X's are uncorrelated each other to begin with, and hence modal decomposition is not necessary.

Except for these two extreme cases, the decreasing feature of eigenvalues can usually be observed for covariance matrices. An example of this is demonstrated below for a two-dimensional isotropic homogeneous field that is discretized into 100 finite elements as shown in Fig. 2. The following analytical forms of the spectral density function and auto-correlation function are assumed for this purpose without loss of generality

$$S_{XX}(\mathbf{K}) = \sigma^2 \frac{d^2}{4\pi} \exp\left[-\left(\frac{d|\mathbf{K}|}{2}\right)^2\right]; \qquad R_{XX}(\xi) = \sigma^2 \exp\left[-\left(\frac{|\xi|}{d}\right)^2\right] \quad \dots\dots (40)$$

in which d is a parameter representing the scale of correlation, $\mathbf{K} = [K_x K_y]^T$ is a wave number vector, and $\xi = [\xi_x \xi_y]^T$ is a separation vector.

The covariance matrix $\mathbf{C}_{XX}$ is constructed using Eqs. 2 and 40 for the finite element model. Ordinarily, the mesh discretization must be examined to see

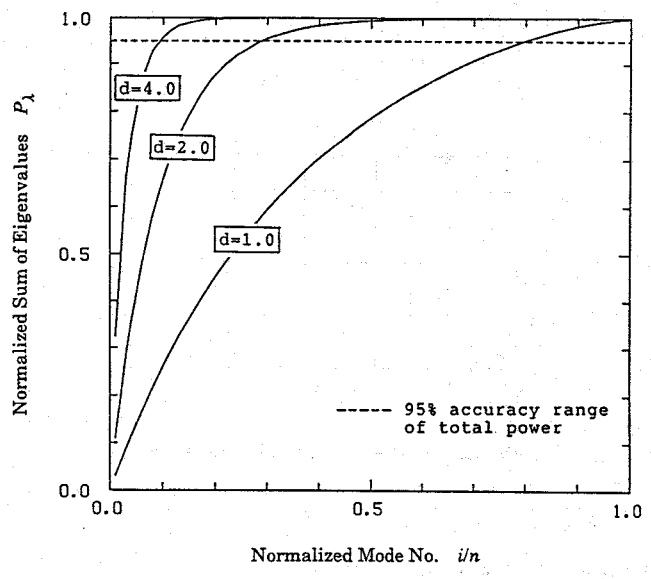

FIG. 3. Relationship between Normalized Mode Number and Normalized Sum of Eigenvalues

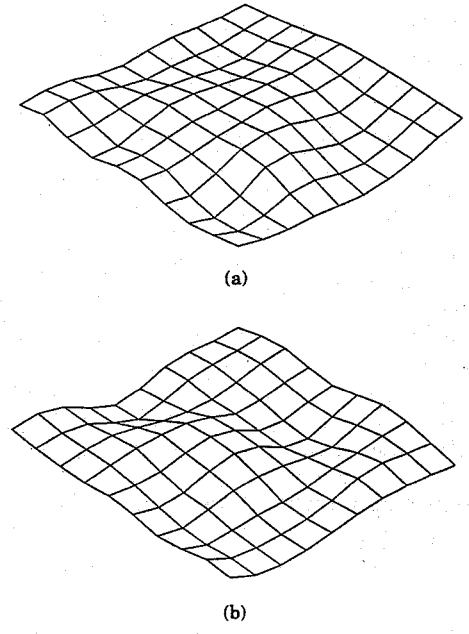

FIG. 4. Simulated Stochastic Fields: (*a*) Statistical Preconditioning; (*b*) Random Sampling

whether or not it is small enough to represent the random variabilities. Thus, the mesh discretization depends on the scale of correlation presently represented by the parameter d; the larger d, the more slowly the correlation disappears as a function of the separation distance. However, the same element size $l_x = l_y = 1.0$ is used for three values of d, (d = 1.0, 2.0, and 4.0) in this study to observe how the eigenvalues λ_i decrease in value as i increases. In this respect, the reader is reminded of the earlier comment that the primary purpose of this study is to develop a state-of-the-art method to simulate and digitally generate discretized stochastic fields, but not to deal with the element size issue.

The relationship between the normalized mode number i/n and normalized sum P_λ of the eigenvalues up to the ith mode is shown in Fig. 3 for the three values of d. Here, P_λ is defined by

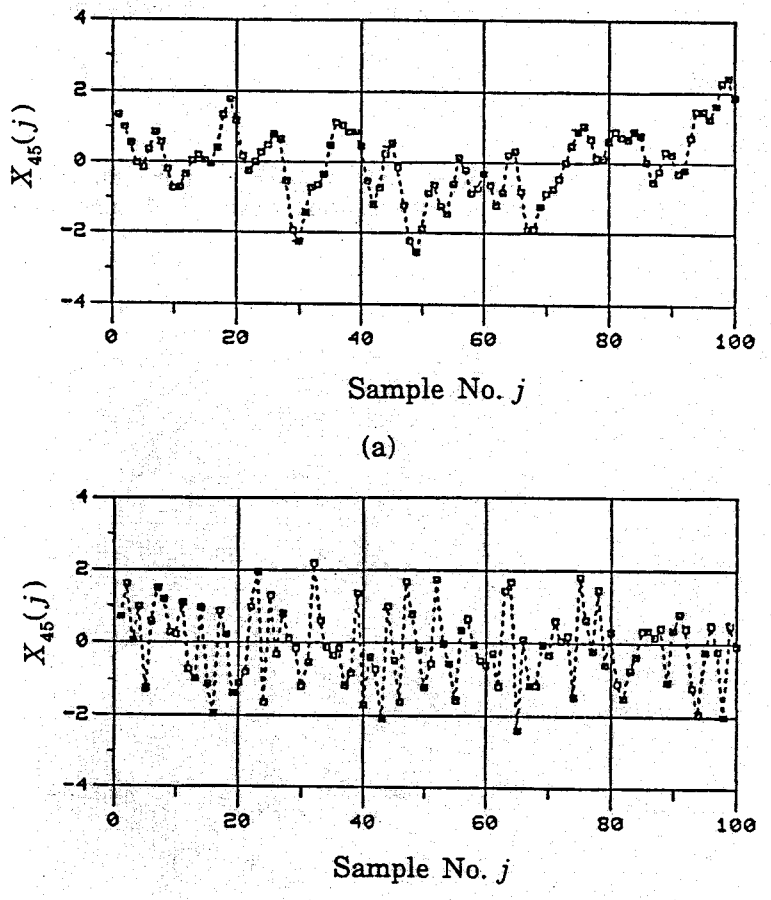

FIG. 5. Simulated Random Processes at Element No. 45: (*a*) Statistical Preconditioning; (*b*) Random Sampling

$$P_\lambda(i) = \frac{\sum_{k=1}^{i} \lambda_k}{\sum_{k=1}^{n} \lambda_k} = \frac{\sum_{k=1}^{i} \lambda_k}{(n\sigma^2)} \quad \text{(41)}$$

It is clearly observed that the eigenvalues associated with the lower modes are more significant than those with higher modes. This trend is more prominent when the ratio l/d of the element length $l = l_x = l_y$ to the correlation distance d is small.

If the finite element discretization is considered strictly from the viewpoint of dealing with material property or geometry variability, the ratio l/d must, in general, be small. Hence, it can be expected that the modal truncation will reduce the number of Z's that must be generated, depending of course on the type of correlation functions that govern the stochastic nature of the field.

Numerical Example

A numerical example is given in order to demonstrate the efficacy of the proposed statistical preconditioning simulation technique. The finite element discretization shown in Fig. 2 and the auto-correlation function represented by Eq. 40 with constants $\sigma = 1.0$ and $d = 2.0$ are used.

The sample size used is $N_s = 2MN_f$ (Eq. 31) by choosing $M = 50$ (number of modes to be superposed) and $N_f = 1$ (number of frequencies in each mode). This truncation represents more than 99% accuracy of the total power of the spatial variation. The ordinary simulation method based on modal decomposition and random sampling of the Z's is also carried out using the same sample size. The results of these two simulation methods are compared next.

Two arbitrary sample stochastic fields generated by the two methods are shown in Fig. 4. There is no way in which the effect of preconditioning can be detected from these figures. However, a clear difference can be observed in Fig. 5 where the random variable $X_{45}(j)$ at element No. 45 is plotted as a function of the sample number j. The random variables generated by the preconditioning method show periodic variations because of the trigonometric functions they involve. On the contrary, the random variables generated by the random sampling method behave more like a white noise.

The target covariance matrix is constructed from $R_{XX}(\xi)$ in Eq. 40 and graphically shown in Fig. 6 where the value of each element of the matrix is represented by the height at the corresponding coordinates of the element centroid in the three-dimensional plot. Similar figures for the sample covariance matrices are shown in Figs. 7 and 8. The covariance matrix gen-

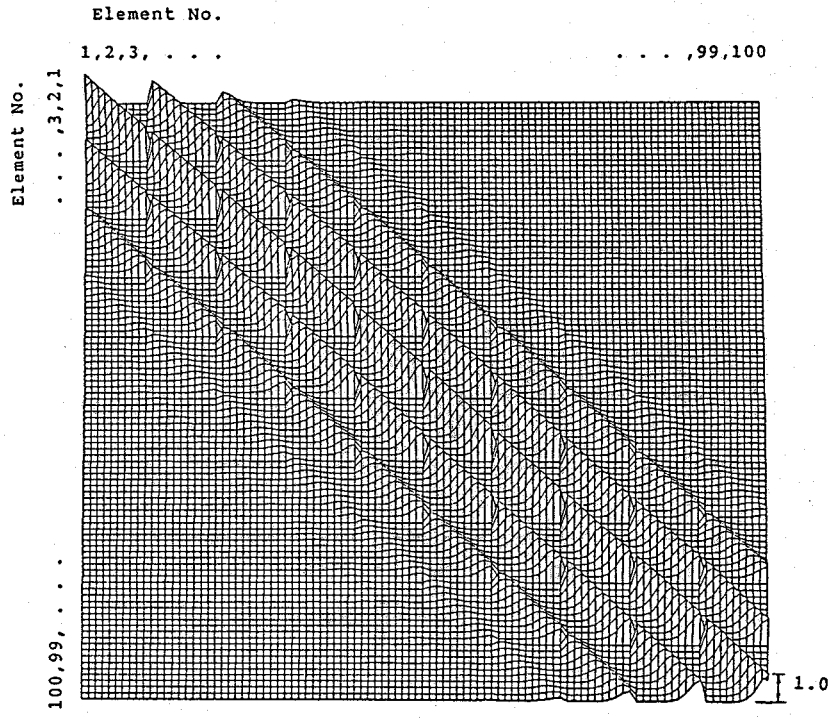

FIG. 6. Target Covariance Matrix

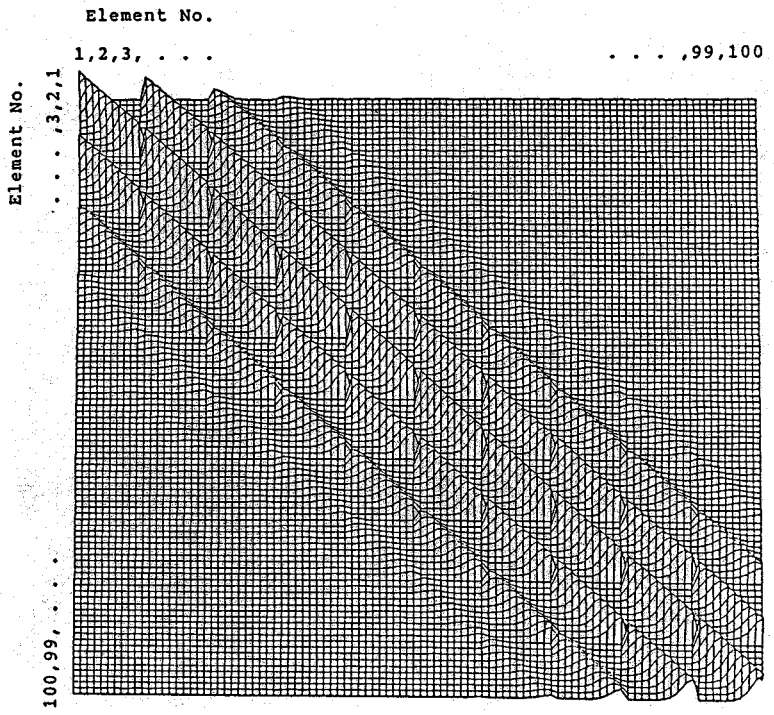

FIG. 7. Simulated Covariance Matrix by Statistical Preconditioning

erated by the preconditioning method is in perfect agreement with the target as expected. However, the covariance matrix generated by random sampling is quite erratic although the same sample size is used. A clearer demonstration of the same fact is shown in Fig. 9 where the sample covariance values along the straight line between the centroids of element Nos. 1 and 100 are plotted. The covariance values based on the sample generated by statistical preconditioning are indicated by square boxes at the separation distances $\xi_{1,12} = 1.414$, $\xi_{1,23} = 2.828$, ..., $\xi_{1,100} = 12.73$ where $\xi_{i,j}$ = distance between the centroids of the ith and jth elements. These square boxes are precisely on the solid curve representing the target spatial correlation function. On the other hand, the sample covariance values based on random sampling exhibit considerable discrepancy from the target values. In fact, a much larger sample size would be required for the random sampling method to produce target covariances acceptable in approximation. This indeed has been one of the major sources of discouragement for the use of Monte Carlo methods. The statistical preconditioning technique presented in this study substantially alleviates this difficulty.

The sample distributions by the two methods are also compared with the target distribution, which is Gaussian. One hundred sample values at element No. 45 are plotted on a Gaussian probability paper in Fig. 10. The goodness of fit of the random variables $X_{45}(j)$ (j = 1, 2, ..., 100) generated by both

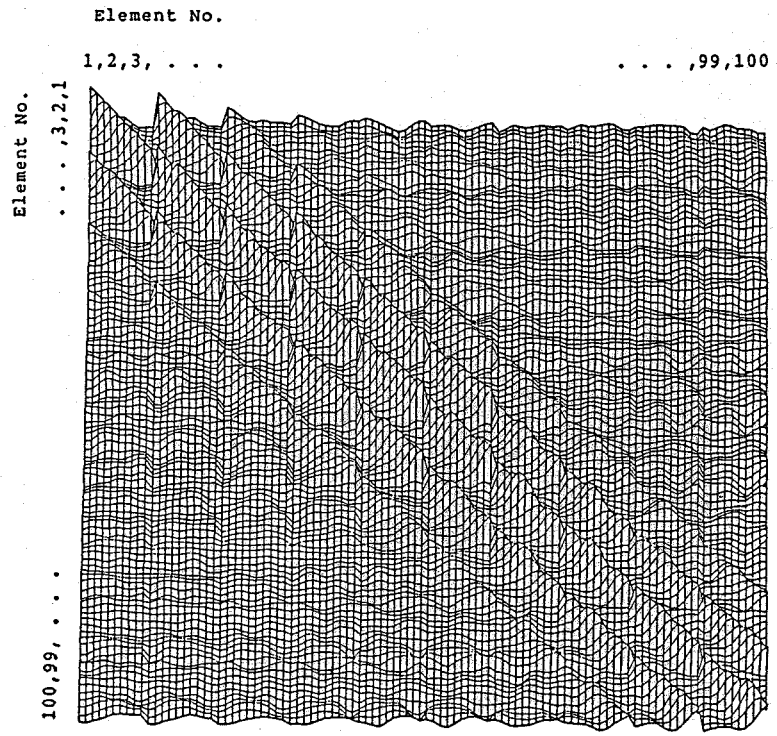

FIG. 8. Simulated Covariance Matrix by Random Sampling

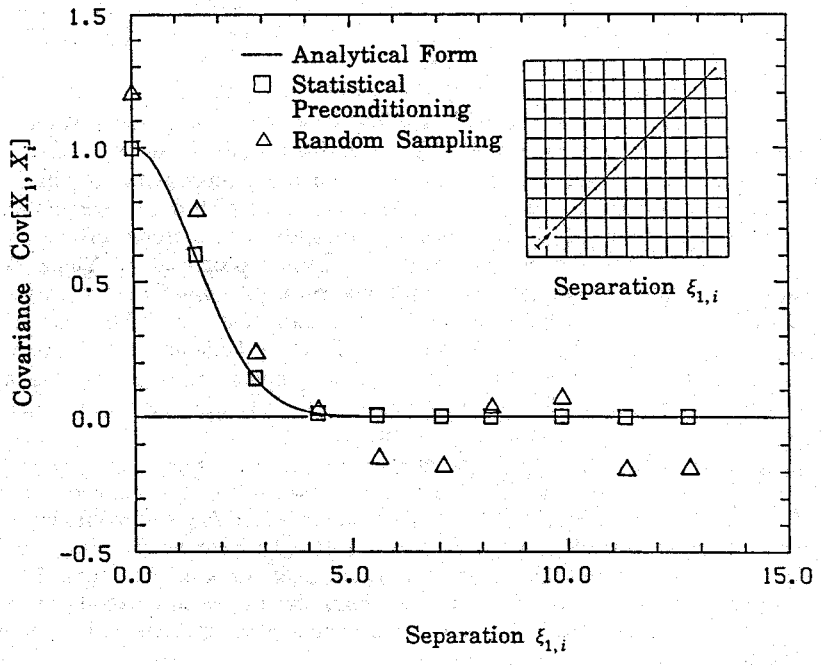

FIG. 9. Analytical and Simulated Covariances between Element Nos. 1 and i

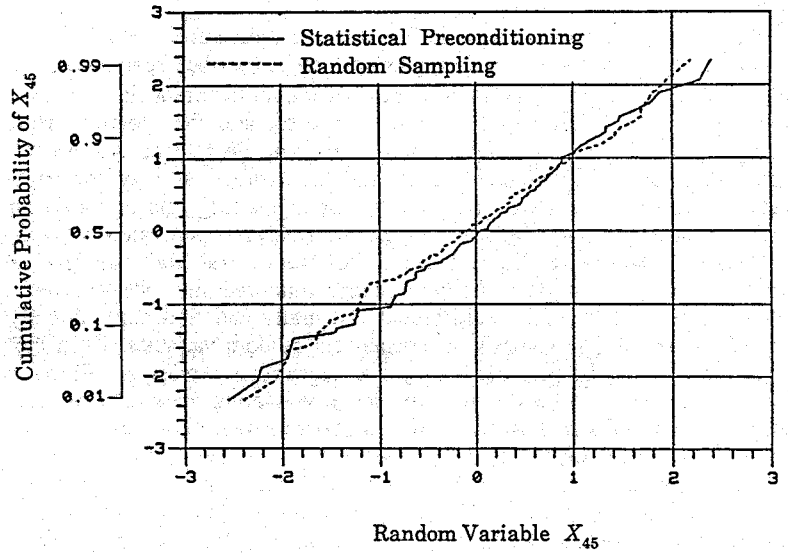

FIG. 10. Statistical Distributions of Simulated Random Variable X_{45}

methods to the target Gaussian distribution with mean-zero and unit standard deviation is about the same for both cases. If a larger sample size is selected, the simulated distributions will further approach a Gaussian distribution, i.e., a straight line on the probability paper.

Although only an example for a univariate, two-dimensional stochastic field is provided herein, the wide applicability of the proposed method is expected. In fact, since the modal decomposition method can be applied to multivariate and multidimensional and even nonhomogeneous fields with little additional effort, the proposed method can deal with digital generation of sample functions of Gaussian stochastic fields with extreme efficiency. Once the Gaussian field is generated, the method proposed by Yamazaki and Shinozuka (1988) can be used to transform it into non-Gaussian fields.

Note that the only difference between the statistical preconditioning and the random sampling lies in use of Eq. 21 or use of Gaussian random numbers. Since the other steps (e.g., the modal decomposition) of the two methods are common, the CPU (central processing unit) time for sample generation is almost the same for both methods.

In passing, it is noted that the point estimate method can generate a sample of a random variable so as to reproduce the first few moments exactly. The method, however, appears to be of little value in generating sample functions of stochastic vectors. In this connection, the Latin hypercube sampling technique can deal with stochastic vectors, although it is not easy to control the cross terms of the covariance matrix. Hence the proposed method can be considered as an extended version of these statistical approaches with a wider applicability.

Conclusion

A method of generating sample functions of a discretized stochastic field with mean zero, which precisely reproduces the prescribed spatial correlation function, is developed. The method utilizes modal decomposition of the covariance matrix of the correlated random vector and the spectral (trigonometric) representation of random processes. The decreasing feature of the eigenvalues of the covariance matrix and the orthogonality of the trigonometric functions are taken advantage of for generating sets of independent random variables. The discretized stochastic field thus generated is Gaussian by virtue of the central limit theorem. For the central limit theorem to be valid, however, a number of trigonometric functions must be summed up. In the present study, this requirement is usually satisfied automatically in the process of modal synthesis of independent random variables. Even though this synthesis may be performed only over significant modes, it still involves a number of terms to be summed up, hence validating the use of the central limit theorem. This makes it possible to generate sample functions of a discretized stochastic field with mean zero, which precisely reproduces, when ensemble-averaged, the prescribed covariance function with a substantially smaller sample size than that required by standard random sampling techniques. The proposed statistical preconditioning technique, therefore, will make the application of Monte Carlo techniques to civil engineering and particularly to earthquake engineering problems involving stochastic fields even more practical.

Acknowledgment

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